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Probability Densities for Noisy Delay Bifurcations

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The delay of a transition in a nonlinear system due to a slowly varying control parameter can be significantly reduced by very small noise. A new asymptotic approximation for the time-dependent probability density function gives a complete description of the process into the transition region, and is easily interpreted in terms of the noisy dynamics. It is also used to calculate mean transition times. The method is applied to two nonlinear systems with noise: a one-dimensional canonical model for a steady bifurcation and the noisy FitzHugh–Nagumo model.

KEY WORDS: Probability density; Fokker–Planck; delay bifurcation; FitzHugh–Nagumo; noise; slowly varying.

1. INTRODUCTION

It is well known that the slow variation of a bifurcation parameter through a critical point can delay the transition to a new state.[1–5] Then the transition takes place at a value of the bifurcation parameter which is far from the critical point. This delay is attributed to a memory effect, in which the system "remembers" the previous state that it was in. Several studies have shown that external oscillations or noise can reduce or eliminate the delay in the transition to the new state, thus affecting this memory of the system.[6–12] In particular, small oscillations have been shown to advance the transition significantly in both steady[8] and Hopf bifurcations,[9] as well as in passing through a resonance.[12] Quantitative predictions about these transitions provide a characterization of the sensitivity of these systems to noise. The probability density function can be used to make these predictions.

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In this paper we study two systems of the form

$$d\mathbf{x} = \mathbf{a}(\mathbf{x}, t)\, dt + \sqrt{2}\, \varepsilon\, d\mathbf{W} \tag{1.1}$$

where $\mathbf{W}$ is a vector of independent Brownian motions, and ε is a measure of the magnitude of the noise. When $\varepsilon = 0$, the system $d\mathbf{x}/dt = \mathbf{a}(\mathbf{x}, t)$ has a delay bifurcation, and for $0 < \varepsilon \ll 1$ this delay is reduced by an $O(1)$ amount. This effect is demonstrated by the simple example of Section 2. Since the dynamical system with noise (1.1) describes a stochastic process $\mathbf{x}$, it is natural to look for the probability density function $p(\mathbf{x}, t)$ since it holds all of the information about the process. The time dependent probability density function is the probability that the process takes a certain value at a certain time,

$$p(\mathbf{x}, t)\, d\mathbf{x} \equiv P(\mathbf{x}(t) \in (\mathbf{x}, \mathbf{x} + d\mathbf{x})) \tag{1.2}$$

This function is used to determine moments of the process $\mathbf{x}$ and the moments of the time until the process makes a transition. The density $p(\mathbf{x}, t)$ satisfies the Fokker–Planck equation (FPE)

$$\frac{\partial p}{\partial t} = \varepsilon^2\, \nabla^2 p - \nabla \cdot (\mathbf{a} p) \tag{1.3}$$

Previous studies look for measures of the effects of the noise in these problems. Even though the probability density function holds all of the information about the process, these studies do not try to find this function for the nonlinear delay bifurcation problems with noise, perhaps because there are several difficulties in solving (1.3). The underlying noiseless process is described by a non-autonomous system, so that there is no steady state solution, and the full time-dependent equation must be solved. In the case of small noise ($\varepsilon \ll 1$) the probability density function can be densely concentrated in one region so that it has sharp gradients. Then numerical methods can be inaccurate or expensive, and it would be preferable to use an asymptotic method based on the small diffusion, such as a WKB expansion.[13] As shown in the following sections, for noisy delay bifurcations the probability density function can vary in shape over time. For some time intervals it is sharply peaked and at other times it is concentrated over a larger region and the gradients are smaller. Then it is necessary to determine when and where the asymptotic methods are valid.

In this paper we approximate the probability density for delay bifurcation problems with small additive noise. The method is based on a Gaussian-type ansatz, similar to that used for the invariant probability density for a chaotic system.[14] In this paper we interpret the behavior of

the density in terms of the dynamics of the system and use the approximation to find the time at which the transition takes place, since it is valid into the transition region. The method has some useful features which are worth emphasizing.

1. The form of approximation for the density reduces the problem from that of solving the partial differential equation (pde) (1.3) to that of solving a system of ordinary differential equations (ode's). Thus the computation is much faster.

2. From the form of the density, which is of Gaussian-type, one can easily interpret the density in terms of the location of the transition region. This information is directly related to the region where the probability density is valid, so that it is clear where the asymptotic approximation can be applied.

3. The asymptotic approximation gives the dependence of the dynamics on both the size of the noise and the other parameters in the model, such as the slow variation of the bifurcation parameter. Then the parametric dependence of the transition times, computed using this density function, is known also.

We briefly illustrate these points with the form of the density for the one-dimensional example studied in Section 2,

$$p(y, t) \sim C \sqrt{g(t)}\, e^{-(y - F(t) - (2\varepsilon/\sqrt{g_n}))^2 (g(t)/2\varepsilon^2)} \tag{1.4}$$

Here y is the stochastic process, $F(t)$ is the solution for the deterministic problem (e.g. (1.1) with $\varepsilon = 0$), g_n and $g(t)$ are functions that are determined by substituting (1.4) into the FPE, and C is a normalizing constant. This substitution yields an ode for $g(t)$, which naturally includes the parameters of the original model. The function $(g(t)/\varepsilon^2)^{-1}$ is analogous to the variance for a normal random variable, so if we can find $g(t)$ we can see how much the process y varies about the perturbed deterministic behavior described by $F(t) + 2\varepsilon/\sqrt{g_n}$. When the function $g(t)/\varepsilon^2$ is small the density no longer has sharp gradients. The form of the approximation is based on the assumption that the probability is densely concentrated in some region, so that the validity of the approximation is limited to values of $g(t)$ which are not too small. In addition to giving insight into the dynamics of the process, the approximation for the density is used to calculate the mean transition time for the process.

We apply the method to two different models with delay bifurcations. In Section 2 we demonstrate the method for a one-dimensional model which has a steady bifurcation. In Section 3 we apply the method to the FitzHugh–Nagumo model which has a Hopf bifurcation. In both cases we

approximate the probability density for times into the transition regions, interpret the density in terms of the dynamics of the process, and use the density to calculate the expected time of transition. The asymptotic approximation is compared with the numerical solution of (1.3) to demonstrate its validity into the transition region. In this paper we focus on the use of the asymptotic approximation, and refer the reader to ref. 15 where the computational issues for solving (1.3) are discussed in detail.

We conclude this section by mentioning related work. The effect of noise on the delay bifurcation is related to the phenomenon of stochastic resonance, where additive noise in a subthreshold excitable system can cause a transition to super-threshold behavior (see ref. 16 and references therein). Quantities such as signal-to-noise ratio, power norms, which involve the mean and variance of super-threshold incidents, and power spectral densities are used to measure the effectiveness of the noise in this phenomenon. The probability density for the dynamics, which are of the type studied in this paper, could be used to calculate these measures of stochastic resonance.

The method used in this paper differs from that of the WKB-type expansions used in refs. 17 and 18, for example, where substituting $p = e^{\psi/\varepsilon^2}$ into the FPE yields the Hamilton–Jacobi-type equation

$$\psi_t = \varepsilon^2 \nabla^2 \psi + |\nabla \psi|^2 - \mathbf{a} \cdot \nabla \psi - \varepsilon^2 \nabla \cdot \mathbf{a} \tag{1.5}$$

The method of this paper is, in effect, solving (1.5) approximately with an appropriate quadratic function for ψ based on the underlying deterministic dynamics. The advantage of the approximation (1.4) is that it is easily interpreted in terms of the dynamics of the system and the density can be found with fast, straightforward computations. A limitation of (1.4) is that it is not valid for all regions of space and time, but it accurately describes the effect of the noise on the transitions in these problems. This is the most interesting part of the dynamics, since after the transition the small noise has very little effect on the dynamics.

Probability densities and mean transition times have been computed for linear models with noise.[6] The models which are considered here are nonlinear. Even though they can be approximated by linear models in some regions, we do not use linear approximations, since this would introduce errors which would obscure the effects of the noise.

2. THE ONE-DIMENSIONAL MODEL WITH A STEADY BIFURCATION

We consider the one-dimensional problem,

$$dy = (\mu t y - y^3)\, dt + \sqrt{2}\, \varepsilon\, dW, \qquad y(t_0) = y_0, \quad 0 < \varepsilon \ll \mu \ll 1 \tag{2.1}$$

which models the effect of very small noise on the delay in steady bifurcations, for W a standard Brownian motion. We review the behavior of the solution of the deterministic problem

$$\frac{dy}{dt} = \mu t y - y^3, \qquad y(t_0) = y_0 \tag{2.2}$$

This equation has been studied in ref. 8, as a canonical model of the delay of a steady bifurcation which occurs in lasers. Many models with similar behavior have been discussed in refs. 19 and 20. If t is replaced by a constant T_0 in the right hand side of (2.2), there are two steady state solutions $y_1 = 0$ for all T_0 and $y_2 = (\mu T_0)^{1/2}$ for $T_0 > 0$. For $T_0 > 0$ the solution y_2 is stable. That is, there is a steady bifurcation from the zero solution y_1 to the solution $y_2 = (\mu T_0)^{1/2}$ at the critical parameter value $T_0 = T_c \equiv 0$.

Now consider (2.2), where t is the time variable. For values of $t \ll 0$, $y(t)$ decays exponentially and for $t > 0$, $y(t)$ approaches $y(t) = \sqrt{\mu t}$, so that the value $t = 0$ plays the role of a critical point. For $t_0 < t < 0$ the solution $y(t)$ approaches zero, and as t increases so that $t > 0$ there in an $O(1)$ interval of time during which $|y(t)| \ll 1$. At a time such that $\sqrt{\mu t} = O(1)$, $y(t)$ grows rapidly, eventually approaching $y(t) \sim \sqrt{\mu t}$. This is known as a delay bifurcation, since as the coefficient μt varies slowly through the critical point $t = 0$, the solution does not immediately make the transition to the state $y(t) \sim \sqrt{\mu t}$.

The explicit solution for (2.2) with $\varepsilon = 0$ is

$$y^2(t) = \frac{e^{\mu t^2}}{2 \int_{t_0}^{t} e^{\mu t'^2}\, dt' + C} \equiv F^2(t) \tag{2.3}$$

where $C = e^{\mu t_0^2}/y_0^2$ is determined by the initial condition $y(t_0) = y_0$. This solution has the behavior described above, that is, that $y(t)$ approaches 0 for $t < 0$ and $y(t) \sim \sqrt{\mu t}$ for large t. Figure 1 shows the solution of (2.2) for $y_0 \ll 1$ and $t_0 = 0$, which shows the same delay in transition to $y(t) \sim \sqrt{\mu t}$ as in the case of $y_0 = O(1)$, $t_0 < 0$. Then without loss of generality, in the following we take $t_0 = 0$ and $y_0 \ll 1$ for convenience,

Next we look for the probability density function, which describes the effect of noise on the delay bifurcation. Previous studies considered the reduction of the delay with the addition of sinusoidal oscillations.[8] Numerical simulations of (2.1), as in ref. 21, demonstrate that the delay can be reduced with the addition of noise (see Fig. 1).

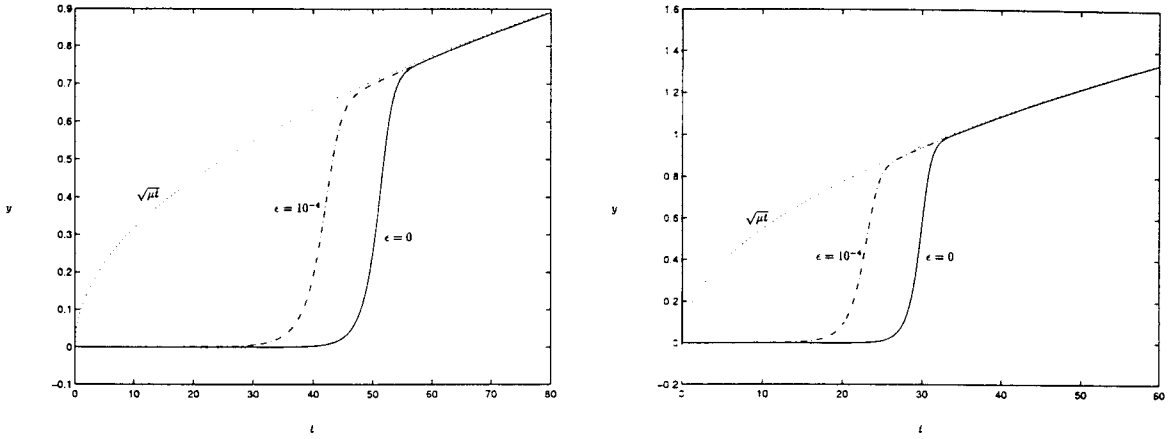

Fig. 1. The solid line is the solution (2.1) with $\varepsilon = 0$ given by (2.3) and the dash-dotted line is the solution of (2.1) with $\varepsilon = 10^{-4}$. The initial condition for both is $y(0) = 10^{-6}$. In (a) $\mu = 0.01$ and in (b) $\mu = 0.03$. Note that the noisy solution makes the transition to a solution near $\sqrt{\mu t}$ (the dotted line) much sooner than the deterministic solution does, that is, the delay of the transition is reduced. The equation is symmetric in y, so that a random realization might also make the transition to $y \sim -\sqrt{\mu t}$.

2.1. The Asymptotic Expansion

The probability density function $p(y, t)$ for the process y described in (2.1) is the solution of the FPE

$$p_t = \varepsilon^2 p_{yy} - ((\mu t y - y^3)\, p)_y, \qquad p(y, 0) = p_0(y), \quad \varepsilon \neq 0 \tag{2.4}$$

Expecting that the noisy dynamics will be close to the deterministic dynamics for $\varepsilon \ll 1$, we make the substitution

$$\varepsilon\eta = y - F(t), \qquad p(y, t)\, dy = \tilde{p}(\eta, t)\, d\eta \tag{2.5}$$

which transforms (2.4) into

$$\begin{aligned}\tilde{p}_t = &-\frac{1}{\varepsilon}\left[\mu t F(t) - F^3(t) - F'(t)\right] \tilde{p}_\eta + \tilde{p}_{\eta\eta} \\ &- (\mu t \eta - 3F^2(t)\, \eta - 3\varepsilon\eta^2 F(t) - \varepsilon^2\eta^3)\, \tilde{p}_\eta - (\mu t - 3(\varepsilon\eta + F(t))^2)\, \tilde{p}\end{aligned} \tag{2.6}$$

The coefficient of $O(\varepsilon^{-1})$ vanishes by the definition of $F(t)$. Then the leading order equation for the density is given by the coefficient of ε^0,

$$\tilde{p}_t \sim \tilde{p}_{\eta\eta} - (\mu t \eta - 3F^2(t)\, \eta)\, \tilde{p}_\eta - (\mu t - 3F^2(t))\, \tilde{p} \tag{2.7}$$

We look for a solution of the form

$$\tilde{p}(\eta, t) \sim \sqrt{g(t)}\, e^{-(\eta - (2/\sqrt{g_n}))^2\, (g(t)/2)} \tag{2.8}$$

Substituting in (2.7) for $\tilde{p}(\eta, t)$, yields an equation for $g(t)$,

$$-\frac{g'(t)}{2} = g^2 + (\mu t - 3F^2(t))\, g \tag{2.9}$$

which gives

$$\begin{aligned} g(t) &= \frac{e^{-2Q}}{2\int_0^t e^{-2Q(t')}\, dt' + c_g} \equiv \frac{g_n(t)}{2\int g_n(t')\, dt'} \\ Q(t) &= \int^t \mu t' - 3F^2(t')\, dt' \end{aligned} \tag{2.10}$$

and

$$p(y, t) \sim \mathscr{C} \sqrt{g(t)}\, e^{-(y - F(t) - (2\varepsilon/\sqrt{g_n}))^2\, (g(t)/2\varepsilon^2)} \tag{2.11}$$

In order to allow for the two possible deterministic behaviors, $y = \pm|F(t)|$, we then use the ansatz

$$p(y,t) \sim C\left[\tfrac{1}{2}\sqrt{g(t)}\, e^{-(y-|F(t)|-(2\varepsilon/\sqrt{g_n}))^2\,(g(t)/2\varepsilon^2)} + \tfrac{1}{2}\sqrt{g(t)}\, e^{-(y+|F(t)|-(2\varepsilon/\sqrt{g_n}))^2\,(g(t)/2\varepsilon^2)}\right] \tag{2.12}$$

Here the initial condition for $p(y,t)$ gives the initial condition for $g(t)$ (and the constant c_g) using (2.11). For example, in our calculations below we take

$$p_0(y) = \frac{1}{\sqrt{2\pi}}\, e^{-(y-y_0)^2/(2\varepsilon^2)}, \qquad y_0 = 10^{-6} \tag{2.13}$$

so that $g(0) = 1$.

The approximation (2.12) to the probability density describes the effects of the noise on the dynamics. From (2.11) we see that the density is sharply peaked about $y = \pm|F(t)| + 2\varepsilon/\sqrt{g_n}$ for $g(t) \gg \varepsilon^2$. Then the process y closely follows the deterministic dynamics $\pm|F(t)|$ plus a perturbation $2\varepsilon/\sqrt{g_n}$. Since $|F(t)| \ll 1$, and in fact $|F(t)| < \sqrt{g(t)}/\varepsilon$ until the transition occurs, the shape of the density appears to be a single peak. The transition occurs when y varies significantly from $\pm|F(t)|$. The function $(g(t)/\varepsilon^2)^{-1}$ describes the spread of the density, or the variation of the dynamics, about $y = F(t) + 2\varepsilon/\sqrt{g_n}$. Thus the behavior of $g(t)$ is directly related to the delay until the transition from $|y| \ll 1$ to $y \sim \pm\sqrt{\mu t}$. A graph of $\sqrt{g(t)}/\varepsilon$ is given in Fig. 2. As t increases, $\sqrt{g(t)}/\varepsilon$ decreases, and the density is less concentrated at $y = F(t) + 2\varepsilon/\sqrt{g_n}$. When $\sqrt{g(t)}/\varepsilon$ is small there is a larger probability of observing stochastic behavior which varies significantly from the deterministic dynamics, that is, $y - F(t) = O(1)$. The range of t for which $\sqrt{g(t)}/\varepsilon$ approaches zero corresponds to the transition region in which y increases rapidly from $y \ll 1$ to $y \sim \sqrt{\mu t}$.

We compare the asymptotic result (2.12) for $p(y,t)$ to a numerical solution of the pde using a gradient particle method.[15] Recall that (2.12) is a Gaussian-type approximation with a time-varying standard deviation $(\sqrt{g(t)}/\varepsilon)^{-1}$. Then, since Gaussian-type forms are generally good approximations when the probability density function is sharply peaked, it is not surprising that the approximation loses accuracy in the tails when $g(t)/\sqrt{\varepsilon}$ becomes too small. Figure 2 suggests that, for $\mu = 0.01$, $\varepsilon = 10^{-4}$, and $y_0 = 10^{-6}$, the asymptotic expansion is valid until some time t which is greater than 30, since $\sqrt{g(t)}/\varepsilon$ approaches zero there. In Fig. 3 we compare the asymptotic result to the numerical solution of the pde (2.4) for $t = 28$ and

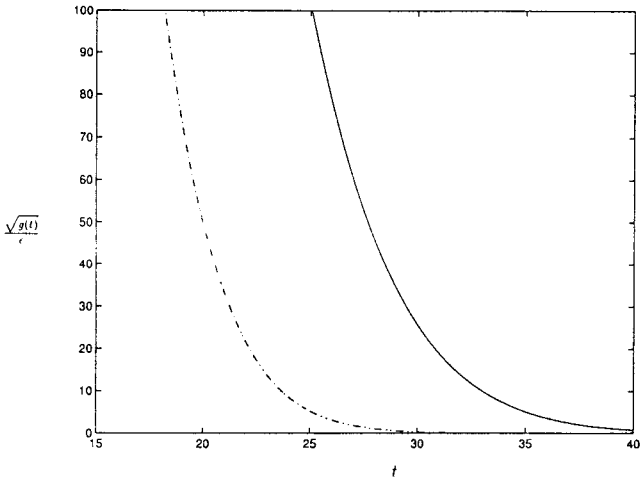

Fig. 2. The graph of $\sqrt{g(t)}/\varepsilon$, which can be interpreted as the inverse of the standard deviation in a Gaussian-type approximation as in (2.12). When this function is large, the probability density is sharply peaked. Then the stochastic dynamics are close to the deterministic dynamics. For small values of this function the density is no longer sharply peaked, that is, the stochastic behavior varies significantly from the deterministic behavior, which corresponds to the reduction in the delay until transition. The dash-dotted line is the result for $\mu = 0.02$, and the solid line is the result for $\mu = 0.01$.

$t = 36$. We see that these results are virtually indistinguishable at $t = 28$, but at $t = 36$ the asymptotic result has a different shape and larger tails than the numerical result. This illustrates that the asymptotic result contains the information about its region of validity; the approximation is valid until $g(t)/\varepsilon^2$ approaches zero. In practice, the application determines the specific range of t for which the asymptotic approximation can be used for the probability density. At the end of this section we use it to compute the expected time until transition, and there we give the range of t used in the computation.

One observation for the behavior of $g(t)$ should be noted. It can be shown from (2.10) that $g(t) \to \infty$ as $t \to \infty$, so that the variation is approaching zero, and the density is strongly peaked about the center. Said another way, as t increases the deterministic dynamics dominate, so that $y(t)$ approaches the deterministic behavior for $\sqrt{\mu t} \gg 1$ and the noise does not play a significant role on these time scales. This is clearly what happens, as one can see from Fig. 1 and the numerical calculation of the probability density for larger values of t in ref. 15. In this example $g(t)$ can be used to indicate long time behavior qualitatively, but in general it can not be used to quantitatively describe the behavior of the density for $\sqrt{\mu t} \gg 1$,

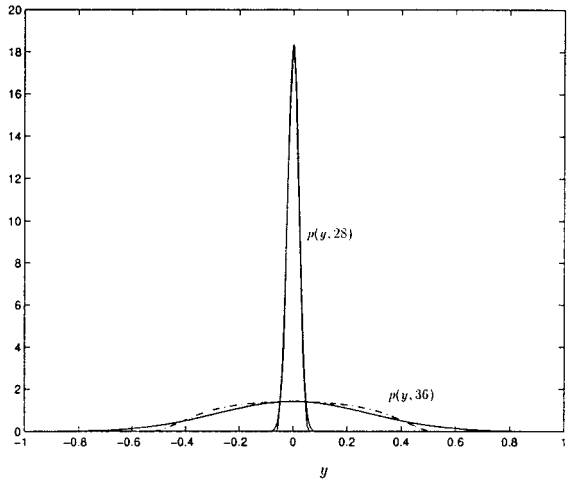

Fig. 3. The probability $p(y, t)$ for $t = 28$ and $t = 36$, with $\mu = 0.01$ and $\varepsilon = 0.0001$. The solid line is the asymptotic approximation (3.6) and the dash-dotted line is the numerical solution of (2.4).[15] Note that the asymptotic approximation differs in the tails of the density for large enough t. This is not unexpected, since in Fig. 2 the function $\sqrt{g(t)}/\varepsilon$ is small for larger values of t, in particular for $t > 30$.

since the asymptotic method is not valid for values of t near the end of the transition region.

Using the asymptotic approximation for $p(y, t)$ we calculate the expected time $\bar{\tau}$ until the process $y(t)$ begins the transition from the solution $y(t) \sim 0$ to $y(t) \sim \sqrt{\mu t}$. For τ the time it takes one realization of $|y(t)|$ to exceed a critical point y_c, $\bar{\tau}$ is the mean of τ over all realizations. This can be expressed in terms of $p(y, t)$ by

$$\begin{aligned} \bar{\tau} &= \int_0^\infty \mathrm{P}(\tau > t)\, dt \\ &= \int_0^\infty \mathrm{P}(|y| < y_c \text{ at time } t)\, dt \\ &\equiv \int_0^\infty \int_{-y_c}^{y_c} p(y, t)\, dy\, dt \end{aligned} \tag{2.14}$$

In Fig. 4 we compare $\bar{\tau}$ as computed from (2.14) with $\bar{\tau}$ from simulations of (2.1). We choose an appropriate value of y_c such that for $|y| \geqslant y_c$, the noise has little effect on the dynamics, and the transition takes place in a deterministic manner. In order to give a quantitative description of y_c, it

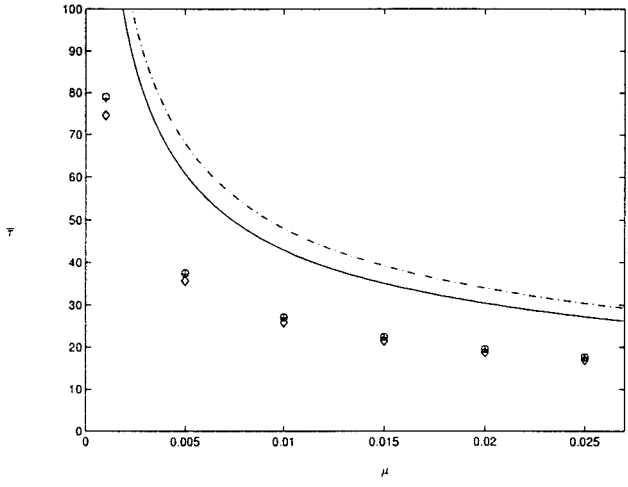

Fig. 4. Comparison of $\bar{\tau}$ for the deterministic and stochastic dynamics. The lines correspond to the deterministic dynamics, where $\bar{\tau}$ is the minimum time such that $y(\bar{\tau}) > 0.01$ (solid line) and $y(\bar{\tau}) > 0.1$ (dash-dotted line) in the deterministic solution (2.3) with initial condition $y(0) = 10^{-6}$. The ○'s and +'s correspond to $\bar{\tau}$, as calculated using (2.14), with $y_c = 0.01$ and $\varepsilon = 0.0001$ for the ○'s, and with $y_c = 0.1$ and $\varepsilon = 0.001$ for the +'s. The ◇'s are the average times until $y > y_c$ from 10,000 realizations of (2.1) with $y_c = 0.01$ and $\varepsilon = 0.0001$. The initial distribution was (2.13).

can be viewed as an initial condition. For $y(0) = y_c$, the difference between the deterministic process and the stochastic process is negligible for all t, so that the probability density is sharply peaked about the deterministic behavior for all time and the function $\sqrt{g(t)/\varepsilon}$ remains large. For the results in Fig. 4, we have chosen y_c such that $\sqrt{g(t)/\varepsilon} > 100$ for all t, so that $y_c = 100\varepsilon$. In Fig. 4 there is good agreement between the calculated value of $\bar{\tau}$ (2.14) and the results from the simulations.

In the expression (2.14) the integral in t is from 0 to ∞. However, the asymptotic expression for $p(y, t)$ is not valid for all time, as discussed above. It is valid for a sufficiently long time $T > \bar{\tau}$ so that contributions to $\bar{\tau}$ from $\int_T^\infty \int_{-y_c}^{y_c} p(y, t)\, dy\, dt$ are negligible, as can be shown computationally. Therefore we replace the upper limit for t in (2.14) with T. The value of T, which is chosen so that it does not affect the calculation of $\bar{\tau}$, varies with μ and ε. For example, for the computations of $\bar{\tau}$ shown in Fig. 4, we used $T = 42$ for $\mu = 0.01$ and $\varepsilon = 0.0001$, and $T = 27$ for $\mu = 0.02$ and $\varepsilon = 0.001$. In general, the value of T was chosen so that increasing T by one would change $\bar{\tau}$ by less than 0.05%. For example, for $\mu = 0.01$ and $\varepsilon = 0.0001$, the difference between $\bar{\tau}$ obtained with $T = 42$ and $\bar{\tau}$ with $T = 43$ is less than 0.005, which is 0.02% of $\bar{\tau} \approx 27$.

3. FITZHUGH–NAGUMO EQUATION WITH NOISE

Now we apply the methods used in the previous section to study the FitzHugh–Nagumo equation with noise,[9, 16]

$$\begin{aligned} du &= b(v-\gamma u)\,dt \\ dv &= (-f(v)-u+I(t))\,dt+\sqrt{2}\,\varepsilon\,dW \end{aligned} \tag{3.1}$$

where a, b, and γ are positive constants, $f(v)=v(v-a)(v-1)$, and W is standard Brownian motion. This system models the propagation of nerve impulses along the giant axon of a squid, and qualitatively captures the features of the Hodgkin–Huxley model. Here v is the potential across the membrane of the axon and u is a recovery current whose behavior follows that of v with a time lag (see, e.g., ref. 10 for further discussion of the model). We study the case in which the applied current increases slowly in time, $I(t)=\mu t$ for $\mu \ll 1$, with small fluctuations in the applied current modeled by $\sqrt{2}\,\varepsilon\,dW$.

We briefly describe the delay in the transition for this model. When the applied current is a constant ($\varepsilon=0$ and $I=$ constant) the system goes to either a steady state, for $I<I_c$, or an oscillatory behavior, for $I>I_c$. The value I_c is the Hopf bifurcation point. When the applied current is slowly increased through this bifurcation point I_c ($I(t)=\mu t$, $\mu \ll 1$, $\varepsilon=0$), the transition from the steady state to the oscillatory state is delayed, that is, it occurs for a value of I significantly larger than I_c. Even when the fluctuations are small, $0<\varepsilon \ll \mu$, the delay in the transition is significantly reduced. Figure 5 compares the deterministic ($\varepsilon=0$) and random ($0<\varepsilon \ll \mu$) solutions of (3.1). Note that the variable v makes the transition to the oscillatory state before u, in both the deterministic and random results.

This reduction in the delay was observed for other types of fluctuations, including sinusoidal oscillations studied in ref. 10 ($I(t)=\mu t+\delta \sin \omega t$), and fluctuations due to numerical noise, studied in ref. 9. In this paper we treat the fluctuations as white noise, which yields (3.1) as a system of stochastic differential equations. Then the corresponding Fokker–Planck equation for the probability density $p(u, v, t)$ is

$$p_t = -(b(v-\gamma u)\,p)_u - ((-f(v)-u+I(t))\,p)_v + \varepsilon^2 p_{vv} \tag{3.2}$$

3.1. The Asymptotic Expansion

As in the study of the one dimensional model of Section 2, we introduce local variables based on the assumption that for $\varepsilon \ll 1$

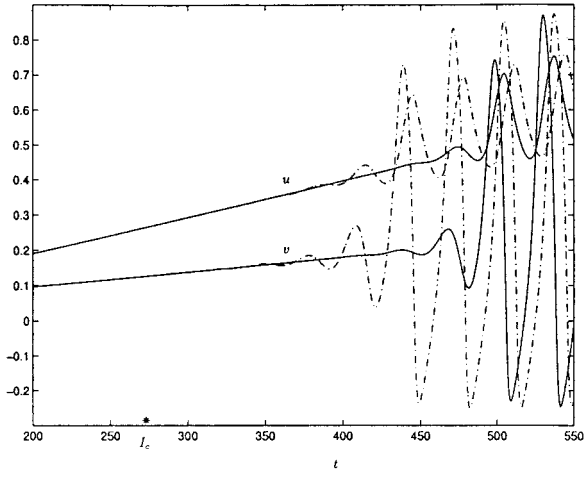

Fig. 5. The solution of (3.1) for $\mu = 0.001$, $a = 0.2$, $b = 0.05$ and $\gamma = 0.4$. The solid line is the deterministic solution ($\varepsilon = 0$) and the dash-dotted line is the noisy result ($\varepsilon = 0.00001$). Note that both transitions occur for $I = \mu t$ above the critical value I_c, but the delay in the case of noisy fluctuations is significantly reduced.

the noisy dynamics will be initially "close" to the deterministic behavior,

$$\zeta \equiv \frac{v - V(t)}{\varepsilon}$$
$$\eta \equiv \frac{u - U(t)}{\varepsilon} \tag{3.3}$$

where $V(t)$ and $U(t)$ are the solution of (3.1) with $\varepsilon = 0$. Substituting (3.3) in (3.2) yields an equation for the transformed joint density $\tilde{p}(\zeta, \eta, t)$,

$$\tilde{p}_t = \tilde{p}_{\zeta\zeta} + (f'(V)\,\zeta + \eta)\,\tilde{p}_\zeta - (b\zeta - b\gamma\eta)\,\tilde{p}_\eta + (f'(V) + b\gamma)\,\tilde{p} + O(\varepsilon) \tag{3.4}$$

using the equations for $U(t)$ and $V(t)$. Then we look for a leading order solution of the form

$$\tilde{p}(\zeta, \eta, t) \sim \mathscr{C} \exp\left[-\frac{g(t)}{2}\zeta^2 - h(t)\,\zeta\eta - \frac{k(t)}{2}\eta^2 + q(t)\,\zeta + r(t)\,\eta + s(t) \right] \tag{3.5}$$

or, in terms of the original variables u and v,

$$p(u, v, t) \sim C \exp\left[-\frac{g(t)}{2\varepsilon^2}(v - V(t))^2 - \frac{h(t)}{\varepsilon^2}(u - U(t))(v - V(t)) \right.$$
$$\left. -\frac{k(t)}{2\varepsilon^2}(u - U(t))^2 + \frac{q(t)}{\varepsilon}(v - V(t)) + \frac{r(t)}{\varepsilon}(u - U(t)) + s(t) \right] \tag{3.6}$$

Substituting (3.6) in (3.4) yields a system of ordinary differential equations for $g(t)$, $h(t)$, $k(t)$, $q(\eta)$, $r(t)$, and $s(t)$. The equations to leading order are

$$\begin{aligned}
-\frac{g'(t)}{2} &= g^2(t) - g(t) f'(V(t)) + bh(t) \\
-h'(t) &= -h(t) f'(V(t)) - g(t) - b\gamma h(t) + 2g(t) h(t) + bk(t) \\
-\frac{k'(t)}{2} &= h^2(t) - h(t) - b\gamma k \\
q'(t) &= -2g(t) q(t) + q(t) f(V'(t)) - br(t) \\
r'(t) &= -2h(t) q(t) + q(t) + b\gamma r \\
s'(t) &= -g(t) + q^2 + f'(V(t)) + b\gamma
\end{aligned} \tag{3.7}$$

In Fig. 6, we show the numerical solution for $g(t)$, $h(t)$, and $k(t)$. The initial condition for these functions is found from the initial condition for $p(u, v, t)$. We take a distribution which is concentrated around $V(0)$ and $U(0)$,

$$p(\zeta, \eta, 0) \sim \frac{1}{2\pi\varepsilon} \exp\left[-\frac{(u - U(0))^2}{2\varepsilon^2} - \frac{(v - V(0))^2}{2\varepsilon^2} \right] \tag{3.8}$$

so that $g(0) = 1$, $k(t) = 1$, $h(0) = q(0) = r(0) = 0$, $s(0) = 0$. Since we start with $q(0) = r(0) = 0$, these terms remain 0.

We solve these equations, focusing on the region in which $g(t)$, $h(t)$, and $k(t)$ approach 0. These coefficients play a role analogous to that of $g(t)$ in Section 2. When these quantities are small, there is a larger probability that the noisy dynamics differ significantly from the deterministic dynamics. That is, the density is less concentrated around the deterministic dynamics when g, h, and k are small.

It is worthwhile to note that these equations are not solved for all time. One could expect from (3.7) that there is singular behavior of the

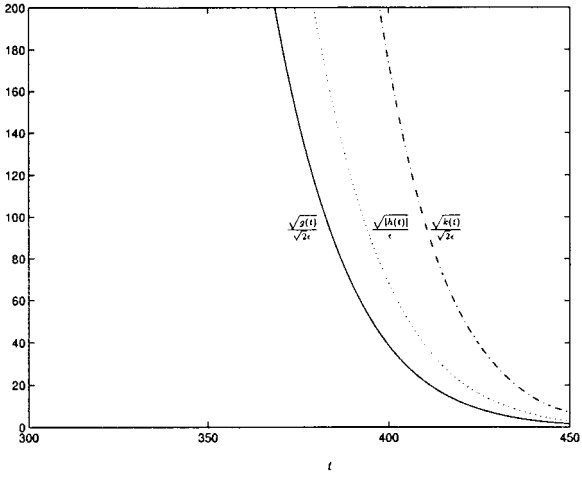

Fig. 6. The solution for $\sqrt{g(t)}/(\sqrt{2}\,\varepsilon)$, $\sqrt{h(t)}/\varepsilon$, and $\sqrt{k(t)}/(\sqrt{2}\,\varepsilon)$ from (3.7), with $\mu = 0.001$, $a = 0.2$, $\gamma = 0.4$, $b = 0.5$. Note that $g(t)$ approaches 0 faster than $k(t)$. This is consistent with the dynamics of (3.1), where $v(t)$ first makes the transition to the oscillatory state, then later u does.

solutions at some time t. In fact this can be observed from the numerical solution of these equations. However, this behavior is observed at times after the interval in which the asymptotic approximation to the density is valid. Therefore this behavior does not affect the approximation.

In Figs. 7 and 8 we compare the asymptotic approximation with numerical results. These results show the change of shape in $p(u, v, t)$ as the process makes the transition from steady behavior to oscillatory behavior. Note that the most dramatic change in $p(u, v, t)$ is in the variable v, which directly reflects that the process v makes the transition to the oscillatory state first and then is followed by u. The comparison with the numerical solution of (3.2) shows that the asymptotic approximation is valid into the transition region.

In Fig. 8 the marginal density function,

$$q(v, t) \equiv \int_{-\infty}^{\infty} p(u, v, t)\, du \tag{3.9}$$

is compared with the numerical result. Even though there is a large change in concentration of $q(v, t)$, there is good agreement with the numerical result well into the transition region. In Section 2, Fig. 3, there is a noticeable difference in the tails of the asymptotic and numerical solutions

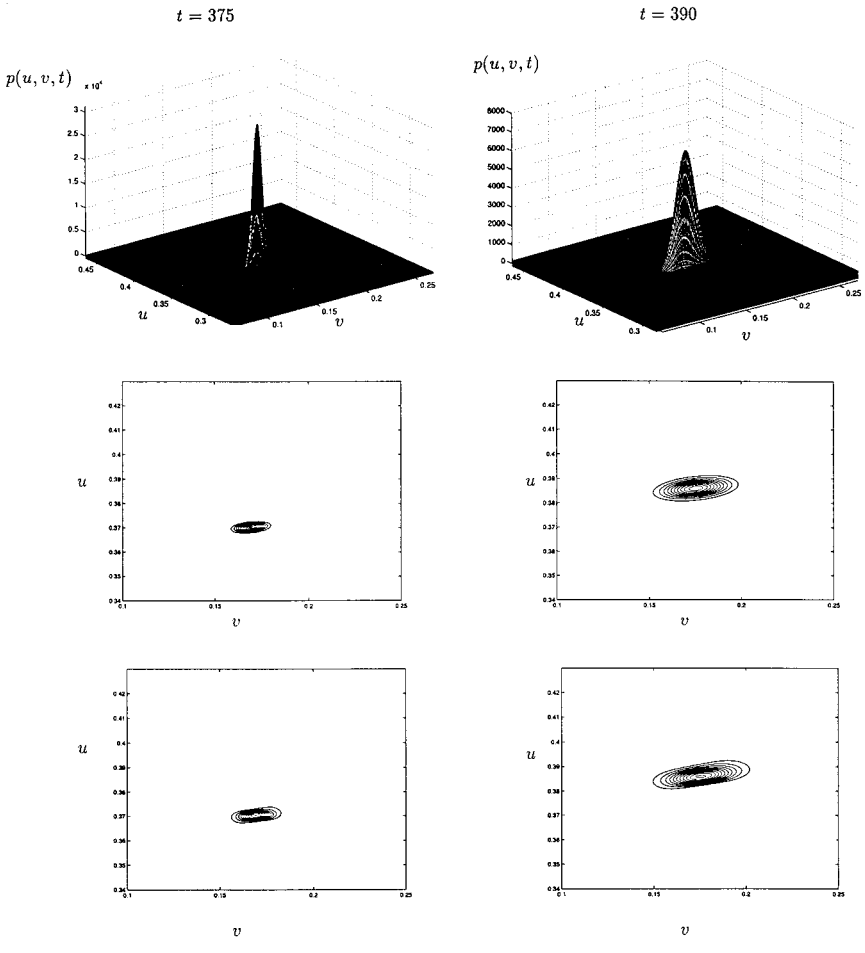

Fig. 7. Comparison of the analytical results for $p(u, v, t)$ with numerical results for $t = 375$ (left column) and $t = 390$ (right column). The top row shows the surface plots of the density (3.6), and it is clear that the shape of the density changes as the transition time approaches. In the middle row are the corresponding contour plots of these density functions. The third row shows the contour plots for the numerically computed solutions for the density functions, which show good agreement with the asymptotic approximation. Here the parameters are $\mu = 0.001$, $\varepsilon = 0.00001$, $a = 0.2$, $\gamma = 0.4$, $b = 0.05$.

for times far into the transition regions. The main difference between the results in Section 2 and Section 3 is the type of transition in the underlying deterministic dynamics. In the one-dimensional example (2.1) the transition from $y(t) \sim 0$ to $y(t) \sim \pm\sqrt{\mu t}$ occurs in such a way that $|y(t)|$ exceeds $\sqrt{\mu t}$ only by the small fluctuations due to the noise. Then the gradient of

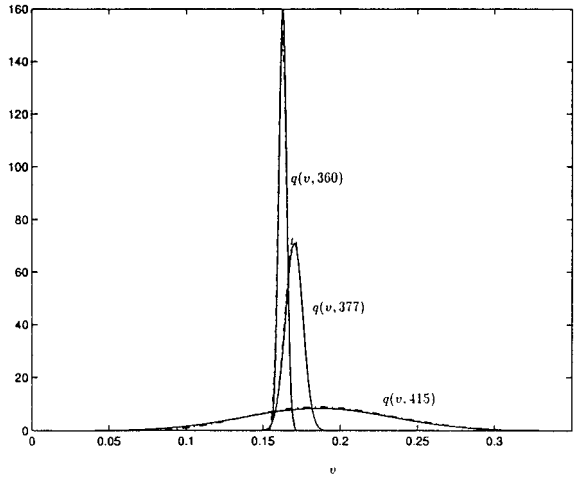

Fig. 8. Comparison of the analytical (solid line) and numerical results (dash-dotted line) for the marginal density $q(v, t)$ for $\mu = 0.001$ and $\varepsilon = 0.00001$, $a = 0.2$, $\gamma = 0.4$, $b = 0.05$. The difference between the asymptotic and numerical results is negligible, even for times well into the transition region.

the density sharpens at $y = \pm\sqrt{\mu t}$ as the density becomes bimodal. In contrast, the transition for the FitzHugh–Nagumo model results in oscillations with increasing amplitude, centered around the (unstable) steady state. Then the density function is symmetric around this steady state, and the symmetry of the Gaussian-type approximation (3.6) gives a better approximation for $p(u, v, t)$ and $q(v, t)$ into the transition region for the two-dimensional model (3.1). The approximation (2.11) also has this symmetry, but it does not reflect the dynamics of the one-dimensional model (2.1) far into the transition region.

As in Section 2, we compute $\bar{\tau}$, the expected time until transition, using the asymptotic result for $p(u, v, t)$. In this application $\bar{\tau}$ is the expected time until the process makes the transition from the steady state to the oscillatory state. The criterion we use for this transition is that the stochastic process $v(t)$ differs by an amount v_c from the deterministic process $V(t)$. The transition is defined only in terms of v since u is virtually slaved to v, with a time lag (see Fig. 5). In terms of the density function,

$$\bar{\tau} \equiv \int_0^\infty \int_{-\infty}^{\infty} \int_{V(t)-v_c}^{V(t)+v_c} p(u, v, t)\, dv\, du\, dt = \int_0^\infty \int_{V(t)-v_c}^{V(t)+v_c} q(v, t)\, dv\, dt \qquad (3.10)$$

Here v_c is chosen sufficiently large so that for $|v - V(t)| > v_c$ the dynamics are not significantly influenced by the noise. As in the one-dimensional

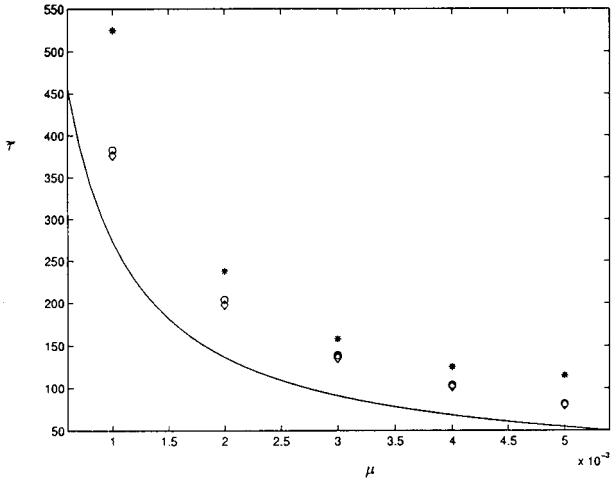

Fig. 9. Comparison of $\bar{\tau}$ for the deterministic and stochastic dynamics of (3.1). The *'s correspond to the deterministic dynamics, where $\bar{\tau}$ is the minimum time such that the oscillations are larger than v_c when $\varepsilon = 0$. The ○'s correspond to $\bar{\tau}$, as calculated using (3.10). The ◇'s give the computed values of the average time until $|v - V(t)| > v_c$ from 40,000 realizations of (3.1). Here $v_c = 0.005$, $\varepsilon = 0.00001$, $a = 0.2$, $\gamma = 0.4$, and $b = 0.05$. The solid line is the critical value I_c, which is the bifurcation point in the case of $I =$ constant.

example, this choice of v_c can be quantified by relating it to the initial condition for the problem where the deterministic and stochastic dynamics remain close, that is, both the stochastic and deterministic solutions make the transition at the same time. Also, the integration in t is truncated at a value T which does not affect the calculation of $\bar{\tau}$, as discussed in Section 2. For example, to calculate $\bar{\tau}$ for $\mu = 0.003$ (see Fig. 9) we replaced the upper limit in (3.10) with $T = 175$. Then increasing T by one changes $\bar{\tau}$ by less than 0.01%. This criterion was also used for the choice of T for the other values of μ. The results for $\bar{\tau}$ as computed from (3.10) are compared to simulations and the deterministic dynamics in Fig. 9.

4. CONCLUSION

In this paper we give an asymptotic approximation for the probability densities for problems with noisy delay bifurcations. In these problems the noise significantly reduces the delay in the transition. In particular we give results for a steady bifurcation problem as well as the noisy FitzHugh–Nagumo model, which has a Hopf bifurcation. The approximation for the probability density is based on a Gaussian-type ansatz which is motivated

by the dynamics. It is easily interpreted in terms of regions of time and space in which the noisy process differs significantly from the deterministic process, and can be used to determine the dependence of the process on the size of the noise, the initial conditions, and other parameters in the problem. The form of the density reveals the regions of validity of the approximation. Since it is valid prior to and during the transition of the process, we use it to give parametric results for the mean transition times of the process.

The ansatz for the density reduces the problem to that of solving a system of ordinary differential equations, which is much less computationally intensive than either solving the the partial differential equation for the density, the Fokker–Planck equation, or simulating many realizations of the stochastic process. The asymptotic approach is general and can be easily applied in other noisy bifurcation problems. It has also been applied to problems with metastable dynamics[22] and in problems where noise changes chaos to nearly periodic behavior.[14] In these problems small noise has an $O(1)$ effect on the dynamics and the probability densities can vary in shape in both time and space.

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